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Equilibrium and least element problems for multivalued functions

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Abstract The principal aim of this paper is to extend some recent results which concern problems involving bifunctions to similar generalized problems for multivalued bifunctions. To this end, by using the appropriate notions of strict pseudomonotonicity we establish the relationships between generalized vector equilibrium problems and generalized minimal element problems of feasible sets. Moreover relationships between generalized least element problems of feasible sets and generalized vector equilibrium problems are studied by employing the concept of Z-multibifunctions.

Keywords Least element problem · Equilibrium problem · Nonlinear problem · Pseudomonotonicity · Z-bifunction · Feasible set

1 Introduction

Cryer and Dempster (see [3]) studied the equivalence of linear complementarity problems, linear programs, least-element problems, variational inequality problems, and minimization

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problems in vector lattice Hilbert spaces. Riddell in [7] extended these results to a certain class of nonlinear monotone maps, called Z-maps, acting from a Banach space to its adjoint space. Schaible and Yao (see [8]) proved the equivalence for these problems by introducing stictly pseudomonotone Z-maps operating on Banach lattices, moreover in [1] the results of Schaible and Yao were been extended to point-to-set maps. Fang and Huang (see [4]) introduced the concept of feasible set for an equilibrium problem with a convex cone, generalized the notion of a Z-map for bifunctions and they derived some equivalence results of equilibrium problems, least element problems and nonlinear programming problems. Recently Fang and Huang (see [5]), assuming a strict pesudomonotonicity and Z-maps assumptions, established relationships between vector equilibrium problems, minimal element problems and least element problems and they obtained a generalized sublattice property of feasible sets for vector equilibrium problems in Banach lattices.

In this paper we establish the relationships between generalized vector equilibrium problems and generalized minimal element problems of feasible sets by using the appropriate notions of strict pseudomonotonicity. Moreover we establish relationships between generalized least element problems of feasible sets and generalized vector equilibrium problems by employing the concept of Z-multibifunctions.

2 Basic facts

Let (X, P) be an ordered Banach space, where X is a real Banach space and P is a pointed convex cone. The partial order $\leq$ on X induced by the pointed cone P is defined by declaring $x \leq y$ if and only if $y - x \in P$ for $x,\ y \in X$. P is called a positive cone on X. Let $K \subseteq X$ be a nonempty, closed and convex set. Let Y be a Banach space and $\Pi(Y)$ be the set of all non empty subsets of Y. Let F be a mapping from $P \times P$ to $\Pi(Y)$. Let $C \subseteq Y$ be a proper closed, convex, and solid cone. We denote with $\mathcal{F}_s$ the feasible set of F with respect to P, defined by

$$\mathcal{F}_{\mathrm{s}} = \{x \in P : F(x, x + y) \subseteq C, \forall y \in P\}.$$

We denote with $\mathcal{F}_{\mathrm{w}}$ the weak feasible set of F with respect to P, defined by

$$\mathcal{F}_{\mathrm{w}} = \{x \in P : F(x, x + y) \nsubseteq -\mathrm{int}\, C,\ \forall y \in P\}.$$

In this paper we consider the following problems:

(I) Generalized strong vector equilibrium problem (in short GSVEP): Find $x \in P$ such that $F(x, y) \subseteq C,\ \forall y \in P$.

(II) Generalized weak vector equilibrium problem (in short GWVEP): Find $x \in P$ such that $F(x, y) \nsubseteq -\mathrm{int}\, C,\ \forall y \in P$.

(III) Generalized least element problem associated with $\mathcal{F}_{\mathrm{s}}$ (in short GSLEP): Find $x \in \mathcal{F}_{\mathrm{s}}$ such that $x \leq y,\ \forall y \in \mathcal{F}_{\mathrm{s}}$.

(IV) Generalized least element problem associated with $\mathcal{F}_{\mathrm{w}}$ (in short GWLEP): Find $x \in \mathcal{F}_{\mathrm{w}}$ such that $x \leq y,\ \forall y \in \mathcal{F}_{\mathrm{w}}$.

(V) Generalized minimal element problem associated with $\mathcal{F}_{\mathrm{s}}$: Find $x \in \mathcal{F}_{\mathrm{s}}$ such that there is no $y \in \mathcal{F}_{\mathrm{s}}$ with $y \neq x$ and $y \leq x$.

(VI) Generalized minimal element problem associated with $\mathcal{F}_{\mathrm{w}}$: Find $x \in \mathcal{F}_{\mathrm{w}}$ such that there is no $y \in \mathcal{F}_{\mathrm{w}}$ with $y \neq x$ and $y \leq x$.

The purpose of this paper is to study the relationship of (I)–(VI).

We first recall the following notations and definitions.

Definition 1 A mapping $F : K \to \Pi(Y)$ is said to be *hemicontinuous*, if for any pair $x, y \in K$ and $\alpha \in [0, 1]$, the mapping $\alpha \to F(\alpha x + (1-\alpha)y, y)$ is continuous at 0^+.

Definition 2 (*see*[6]) Let $F : K \times K \to \Pi(Y)$ and C be a closed, convex, and solid cone. F is said to be C-*quasiconvex*, if F for all $x \in K,\ y' \in K,\ y'' \in K$ and $\alpha \in [0, 1]$, we have

$$F(x, y') \subseteq F(x, \alpha y' + (1-\alpha)y'') + C$$

or

$$F(x, y'') \subseteq F(x, \alpha y' + (1-\alpha)y'') + C$$

Remark 1 We observe that Definition 2 means that $F(x, .)$ for each $x \in K$ is C-quasiconvex (see [2]).

Definition 3 (*see [6]*) Let $F : K \times K \to \Pi(Y)$, $C \subseteq Y$ be a closed, convex, and solid cone. F is said to be *explicitly* C-*quasiconvex*, if F is C-quasiconvex and for all $y' \in K,\ y'' \in K$ and $\alpha \in (0, 1)$, we have

$$F(\alpha y' + (1-\alpha)y'', y') \subseteq F(\alpha y' + (1-\alpha)y'', \quad \alpha y' + (1-\alpha)y'') + C$$

or

$$F(\alpha y' + (1-\alpha)y'', y'') \subseteq F(\alpha y' + (1-\alpha)y'', \quad \alpha y' + (1-\alpha)y'') + C$$

and, in case $F(\alpha y' + (1-\alpha)y'', y') - F(\alpha y' + (1-\alpha)y'', y'') \subseteq \text{int}\, C$ for all $\alpha \in (0, 1)$, we have

$$F(\alpha y' + (1-\alpha)y'', y') \subseteq F(\alpha y' + (1-\alpha)y'', \quad \alpha y' + (1-\alpha)y'') + C$$

Definition 4 Let $F : K \times K \to \Pi(Y)$ and $C \subseteq Y$ be a closed, convex, and solid cone. The multibifunction F is said to be

(i) *pseudomonotone* of Type I if for all $x, y \in K$, we have

$$F(x, y) \not\subseteq -\text{int}\, C \Rightarrow F(y, x) \not\subseteq \text{int}\, C.$$

(ii) *strictly pseudomonotone* of Type I if for all $x, y \in K$, $x \neq y$, we have

$$F(x, y) \not\subseteq -\text{ int}\, C \Rightarrow F(y, x) \subseteq -\text{int}\, C.$$

(iii) *pseudomonotone* of Type II if for all $x, y \in K$, we have

$$F(x, y) \subseteq C \Rightarrow F(y, x) \not\subseteq \text{int}\, C.$$

(iv) *strictly pseudomonotone* of Type II if for all $x, y \in K$, $x \neq y$, we have

$$F(x, y) \subseteq C \Rightarrow F(y, x) \subseteq -\text{int}\, C.$$

Remark 2 The property of (strict) pseudomonotonicity of Type II implies (strict) pseudomonotonicity of Type I.

Definition 5 A multibifunction $F : K \times K \to \Pi(Y)$ is said to be *negative at infinity*, if, for each $x \in K$ there exists some constant $\rho(x)$ such that $F(y, x) \subseteq -\text{int}\, C$ for all $y \in K$ with $\|y\| \geq \rho(x)$.

3 *Z*-map and *Z*-multibifunction

We first recall the following definitions. Let X^* be the adjoint space of X with $\langle u, x\rangle$ denoting the value of $u \in X^*$ at $x \in X$. Let P^* be the dual cone

$$P^* = \{u \in X^* : \langle u, x\rangle \geq 0, \forall x \in P\}$$

Definition 6 (*see [7]*) Let X be a Banach space which is also a vector lattice with positive cone P, and let $T : P \to P^*$. T is said a Z-map relative to P if for any $x, y, z \in P$, $z \wedge (x - y) = 0$ implies that $\langle T(x) - T(y)\rangle \leq 0$.

Definition 7 Let (X, P) be a *vector lattice* induced by a pointed, closed and convex cone P and a bifunction $F : P \times P \to R$. F is said a Z-bifunction if for any $x, y, z \in P$, $z \wedge (x - y) = 0$ and $F(x, x + z) \geq 0$ imply that $F(y, y + z) \geq 0$;

Example 1 Consider $X = R$ and $P = R_+$, and F be defined by

$$F(x, y) = (x^2 - 1)(y - x).$$

It is easy to see that F is a Z-bifunction.

Proposition 1 *Let (X, P) be a vector lattice induced by a pointed, closed and convex cone P and $T : P \to X^*$ be a Z-map. Then the bifunction $F(x, y) = \langle T(x), y - x\rangle$ is a Z-bifunction.*

The following definition extends the definition of Z-map introduced in [7] and Definition 7.

Definition 8 Let (X, P) be a *vector lattice* induced by a pointed, closed and convex cone P and $F : P \times P \to \Pi(Y)$ be a multibifunction. F is said a Z-multibifunction of type I if for any $x, y, z \in P$, $z \wedge (x - y) = 0$ and $F(x, x + z) \not\subseteq -\text{int}\, C$ imply that $F(y, y + z) \not\subseteq -\text{int}\, C$.

Example 2 Let $X = Y = R^2$ and $P = R^2_+$, $F : P \times P \to \Pi(Y)$ be defined by

$$F(x, y) = \left\{(u, v) \in R^2 : u = 1 - y_1, v \in \left[0, x_2^2\right]\right\}$$

Let $x = (x_1, x_2),\ y = (y_1, y_2),\ z = (z_1, z_2) \in P$ with $z \wedge (x - y) = 0$; we have

$$F(x, x + z) = \left\{(u, v) \in R^2 : u = 1 - x_1 - z_1, v \in \left[0, x_2^2\right]\right\}$$
$$F(y, y + z) = \left\{(u, v) \in R^2 : u = 1 - y_1 - z_1, v \in \left[0, y_2^2\right]\right\}$$

We consider the following cases:

(1) $z_1, z_2 \geq 0, x_1 = y_1, x_2 = y_2$

$$F(x, x + z) = \left\{(u, v) \in R^2 : u = 1 - x_1 - z_1, v \in \left[0, x_2^2\right]\right\} \not\subseteq -intC$$
$$\Rightarrow F(y, y + z) = \left\{(u, v) \in R^2 : u = 1 - x_1 - z_1, v \in \left[0, x_2^2\right]\right\} \not\subseteq -intC$$

(2) $z_1 \geq 0, z_2 = 0, x_1 = y_1, x_2 > y_2$

$$F(x, x + z) = \left\{(u, v) \in R^2 : u = 1 - x_1 - z_1, v \in \left[0, x_2^2\right]\right\} \not\subseteq -intC$$
$$\Rightarrow F(y, y + z) = \left\{(u, v) \in R^2 : u = 1 - x_1 - z_1, v \in \left[0, y_2^2\right]\right\} \not\subseteq -intC$$

(3) $z_1 = 0, z_2 \geq 0, x_1 > y_1, x_2 = y_2$

$$F(x, x+z) = \{(u, v) \in R^2 : u = 1 - x_1, v \in [0, x_2^2]\} \nsubseteq -intC$$
$$\Rightarrow F(y, y+z) = \{(u, v) \in R^2 : u = 1 - y_1, v \in [0, x_2^2]\} \nsubseteq -intC$$

(4) $z_1 = z_2 = 0, x_1 > y_1, x_2 > y_2$

$$F(x, x+z) = \{(u, v) \in R^2 : u = 1 - x_1, v \in [0, x_2^2]\} \nsubseteq -intC$$
$$\Rightarrow F(y, y+z) = \{(u, v) \in R^2 : u = 1 - y_1, v \in [0, y_2^2]\} \nsubseteq -intC$$

Thus, F is a Z-multibifunction of type I.

Moreover the following definition generalizes Definition 2.7 of [5] of vector Z-map of type II.

Definition 9 Let (X, P) be a *vector lattice* induced by a pointed, closed and convex cone P and a multibifunction $F : P \times P \to \Pi(Y)$. F is said a Z-multibifunction of type II if for any $x, y, z \in P$, $z = x \wedge y$ and $F(x, x + y - z) \nsubseteq -\text{int } C$ imply that $F(z, y) \nsubseteq -\text{int } C$.

Proposition 2 *Let (X, P) be a vector lattice induced by a pointed, closed and convex cone P and $F : P \times P \to \Pi(Y)$ be a Z-multibifunction of type I. Then F is a Z-multibifunction of type II.*

Proof Let $x, y, z \in P$ such that $z = x \wedge y$. It follows that $(x - z) \wedge (y - z) = 0$.

Let $F : P \times P \to \Pi(Y)$ be a Z-multibifunction of type I. If $F(x, x + y - z) \nsubseteq -\text{int } C$, then $F(z, z + y - z) = F(z, y) \nsubseteq -\text{int } C$. □

4 Main results

In this section we derive some results on the relationships between problems (I)–(VI) under some suitable conditions. In what follows, unless otherwise specified, we assume that P is a pointed, closed and covex cone of a real Banach space X and (X, P) is a lattice.

Theorem 1 *Suppose that $F : P \times P \to \Pi(Y)$ is a strictly pseudomonotone multivalued bifunction of Type I. Then every solution of (II) is also a solution of (VI).*

Proof Let $x^* \in P$ be a solution of GWVEP, this implies

$$F(x^*, y) \nsubseteq -\text{int}C, \quad \forall y \in P. \tag{1}$$

Since P is a cone, one has $F(x^*, x^* + y) \nsubseteq -\text{int}C$ for any $y \in P$, then $x^* \in \mathcal{F}_\text{w}$. We claim that x^* is a solution of GWLEP. Suppose that x^* is not a solution of GWLEP, then there exists some $x \in \mathcal{F}_\text{w}$, with $x^* \neq x$ such that $x^* - x \in P$. Then $F(x, x + x^* - x) \nsubseteq -\text{int}C$ and, by the fact that F is a strictly pseudomonotone multivalued bifunction of Type I, we have $F(x^*, x) \subseteq -\text{int}C$, which contradicts (1). The proof is complete. □

Theorem 2 *Suppose that $F : P \times P \to \Pi(Y)$ is a strictly pseudomonotone multivalued bifunction of Type II. Then every solution of (I) is also a solution of (V).*

Proof Let $x^* \in P$ be a solution of GSVEP, this implies

$$F(x^*, y) \subseteq C, \quad \forall y \in P. \tag{2}$$

Since P is a cone, one has $F(x^*, x^* + y) \subseteq C$ for any $y \in P$, then $x^* \in \mathcal{F}_s$. We claim that x^* is a solution of problem (V). Suppose that x^* is not a solution of problem (V), then there exists some $x \in \mathcal{F}_s$, with $x^* \neq x$ such that $x^* - x \in P$. Then $F(x, x + x^* - x) \subseteq C$ and by the fact that F is a strictly pseudomonotone multivalued bifunction of Type II, we have $F(x^*, x) \subseteq -\text{int}C$, which contradicts (2). The proof is complete. □

Theorem 3 *Suppose that $F : P \times P \to \Pi(Y)$ is strictly pseudomonotone of type I and F is a Z-multibifunction of Type I. Then every solution of (II) is also a solution of (IV).*

Proof Let $x^* \in P$ be a solution of GWVEP, this implies

$$F(x^*, y) \not\subseteq -\text{int}C, \quad \forall y \in P. \tag{3}$$

Since P is a cone, one has $F(x^*, x^* + y) \not\subseteq -\text{int}C$ for any $y \in P$, then $x^* \in \mathcal{F}_{\text{w}}$. Now, we will prove that $x^* \leq x$ for any given $x \in \mathcal{F}_{\text{w}}$. For that, let $z = x^* \wedge x$. Since $x^* \in P$ and $x \in P$ and (X, P) is a vector lattice then $z \in P$. Substituting z into (3), we have

$$F(x^*, z) \not\subseteq -\text{int}C \tag{4}$$

and, by the fact F is pseudomontone, from (4) it follows that

$$F(z, x^*) \subseteq -\text{int}\, C. \tag{5}$$

Moreover, if $x \in \mathcal{F}_{\text{w}}$, as $x^* - z \in P$, $F(x, x + x^* - z) \not\subseteq -\text{int}C$. By the fact that F is a Z-multibifunction of type I, it follows that $F(z, x^*) \not\subseteq -\text{int}C$ which contradicts (5). The proof is complete. □

Theorem 4 *Suppose that $F : P \times P \to \Pi(Y)$ is strictly pseudomonotone of type II and F is a Z-multibifunction of Type I. Then every solution of (I) is also a solution of (III).*

Proof Let $x^* \in P$ be a solution of GSVEP. Then

$$F(x^*, y) \subseteq C, \quad \forall y \in P. \tag{6}$$

Since P is a cone, one has $F(x^*, x^* + y) \subseteq C$, $\forall y \in P$, then $x^* \in \mathcal{F}_{\text{s}}$. Now we will prove that $x^* \leq x$ for any given $x \in \mathcal{F}_{\text{s}}$. Let $z = x^* \wedge x$. Since $x^* \in P$ and $x \in P$ and (X, P) is a vector lattice, we have $z \in P$. Substituting z into (6), we have

$$F(x^*, z) \subseteq C \tag{7}$$

and, by the fact that F is strictly pseudomontone of Type II, from (7) it follows that

$$F(z, x^*) \subseteq -\text{int}\, C. \tag{8}$$

Moreover, if $x \in \mathcal{F}_{\text{x}}$, as $x^* - z \in P$, $F(x, x + x^* - z) \subseteq -\text{int}C$. Since F is a Z-multibifunction of type I, it follows that

$$F(z, x^*) \not\subseteq -\text{int}C. \tag{9}$$

which contradicts (8). The proof is complete. □

5 Properties of feasible sets

Proposition 3 (see [6]) *Let X, Y be real Banach spaces and K be a nonempty, bounded and closed convex subset of X, $F : K \times K \to \Pi(Y)$ and $C \subseteq Y$ be a closed, convex, and solid cone. Assume that*

(i) *F is a pseudomonotone multibifunction of Type I;*
(ii) *$F(x, x) \subseteq C$ for all $x \in K$;*
(iii) *F is explicitly C-quasiconvex;*
(iv) *$F(\cdot, y)$ is hemicontinous for all $y \in K$;*
(v) *$\tilde{F}(y) = \{x \in K : F(y, x) \subseteq int\, C\}$ is open for all $y \in K$.*

Then the GWVEP has solution.

Remark 3 If K is not bounded, a coercivity condition guarantees the existence of a solution ([5]).

Lemma 1 *Let $P \subseteq X$ a closed convex cone, X be a (reflexive) real Banach space, $F : P \times P \to \Pi(Y)$ be a (strictly) pseudomonotone multivalued bifunction of Type I and $C \subseteq Y$ be a closed, convex, and solid cone. Then for a fixed $z \in P$, the multibifunction $F_z : P \times P \to \Pi(Y)$ defined by $F_{rmz}(x, y) = F(x + z, y + z)$, $x, y \in P$ is (strictly) pseudomonotone of type I.*

Proof For $x, y \in P$, suppose $F_z(x, y) \not\subseteq -\text{int}\, C$, then $F(x + z, y + z) \not\subseteq -\text{int}\, C$. As F is pseudomonotone of type I, it follows that $F(y+z, x+z) \not\subseteq \text{int}\, C$ and hence $F_z(y, x) \not\subseteq \text{int}\, C$. Therefore F_z is also pseudomonotone of type I. □

Proposition 4 *Let X be a real reflexive Banach space. Let Y be a real Banach space and C be a closed, convex, and solid cone. Let $P \subseteq X$ be a pointed, closed and convex cone and $F : P \times P \to \Pi(Y)$ be a pseudomonotone multibifunction of Type I and negative at infinity. Suppose that*

(i) *$F(x, x) \subseteq C$ for all $x \in P$;*
(ii) *F is explicitly C-quasiconvex;*
(iii) *$F(x, \cdot)$ is lower semicontinuous for all $x \in P$;*
(iv) *$F(\cdot, y)$ is hemicontinuous for all $y \in P$;*
(v) *for a fixed $z \in P$, for any given $x, y \in P$, $F_z(x, w) \not\subseteq -intC$ implies that $F_z(x, y) \not\subseteq -intC$ with $w = \lambda y + (1 - \lambda)x$, $0 < \lambda < 1$.*

Then for any given $z \in P$,

$$\exists x^* \in P : F(z + x^*, z + y) \not\subseteq -int\, C, \ \forall y \in P. \tag{10}$$

Proof For fixed $z \in P$ the function $F_z(\cdot, y)$ is hemicontinous for all $y \in P$. By Lemma 1, F_z is also pseudomonotone of Type I.

Let $\rho = \|z\| + \rho(z)$ where $\rho(z)$ is defined as in Definition 5. Let

$$D = \{y : y \in P, \|y\| \leq \rho\}.$$

By Proposition 3 there exists a point $x^* \in P$ with $\|x^*\| \leq \rho$ such that

$$F_z(x^*, y) = F(x^* + z, y + z) \not\subseteq -\text{int}C, \ \forall y \in D. \tag{11}$$

Now, we prove that $\|x^*\| < \rho$. In fact $\|x^*\| = \rho$ implies $\|x^* + z\| \geq \|x^*\| - \|z\| = \rho(z)$ and, since F is negative at infinity,

$$F(x^* + z, z) \subseteq -\text{int}C \tag{12}$$

Moreover, letting $y = 0$ in (11), one has $F(x^*+z, z) \not\subseteq -\text{int}C$, which contradicts (12). Therefore $\|x^*\| < \rho$. Given $y \in P$, we can choose $0 < \lambda < 1$ such that $w = \lambda y + (1-\lambda)x^* \in D$.

It follows that

$$F_z(x^*, w) = F(x^* + z, \lambda y + (1-\lambda)x^* + z) \not\subseteq -\text{int}C. \tag{13}$$

Therefore by assumption (v) we have $F_z(x^*, y) \not\subseteq -\text{int}C, \ \forall y \in P$. □

Corollary 1 *Suppose that F satisfies the assumptions of Proposition 4 and in addition F is strictly pseudomonotone of Type I. Then the solution x^* of the problem (10) is unique.*

Now we give a result about a property of the feasible set $\mathcal{F}_w$ of a multivalued Z-bifunction F with respect to P.

Theorem 5 *Let X be a real reflexive Banach space. Let Y be a real Banach space and C be a closed, convex, and solid cone. Let $P \subseteq X$ be a pointed, closed and convex cone and $F : P \times P \to \Pi(Y)$ be a strictly pseudomonotone of Type I Z-multibifunction and negative at infinity. Suppose that*

(i) *$F(x, x) \subseteq C$ for all $x \in P$;*
(ii) *F is explicitly C-quasiconvex;*
(iii) *$F(x, \cdot)$ is lower semicontinuous for all $x \in P$;*
(iv) *$F(\cdot, y)$ is hemicontinuous for all $y \in P$.*
(v) *for any given $z \in P$, $F_z(x, w) \not\subseteq -\text{int}C$ implies $F(x, y) \not\subseteq -\text{int}C$ with $w = \lambda y + (1-\lambda)x^*$, $0 < \lambda < 1$, $x, y \in P$.*

Then the feasible set F is a sublattice, i.e., $x \in \mathcal{F}_w$ and $y \in \mathcal{F}_w$ imply that $x \wedge y \in \mathcal{F}_w$.

Proof Suppose $x, y \in \mathcal{F}_w$ and let $z = x \wedge y$. Since $x, y \in P$, also $z \in P$. By Proposition 4, there exists $x^* \in P$ such that

$$F(x^* + z, y + z) \not\subseteq -\text{int}\, C, \ \forall y \in P. \tag{14}$$

For any $v \in P$, $x^* + v \in P$ thus $F(x^* + z, x^* + v + z) \not\subseteq -\text{int}\, C, \ \forall v \in P$.

Let $z_1 = x \wedge (z + x^*)$ and $z_2 = y \wedge (z + x^*)$. Suppose $z_1 \neq z + x^*$. By the fact $x \in \mathcal{F}_w$ and $z + x^* - z_1 \in P$, we have $F(x, x + z + x^* - z_1) \not\subseteq -\text{int}\, C$.

From the definition of a Z-multibifunction, it follows that $F(z_1, x^* + z) \not\subseteq -\text{int}\, C$.

By using the strictly pseudomonotonicity of F, we have that

$$F(x^* + z, z_1) \subseteq -\text{int}\, C. \tag{15}$$

Moreover, substituting $z_1 - z$ into (14) we have $F(x^* + z, z_1) \not\subseteq -\text{int}\, C$ which contradicts (15). Then $z_1 = z + x^*$ and so $z + x^* \leq x$. Similarly, we can prove that $z_2 = x^* + z$ and $x^* + z \leq y$. Hence $x^* + z \leq x \wedge y = z$ and so $x^* = 0$. The proof is complete. □

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